Solving an Optimization Problem using Implicit Differentiation

Suppose you wish to build a grain silo with volume V made up of a steel cylinder and a hemispherical roof. The steel sheets covering the surface of the silo are quite expensive, so you wish to minimize the surface area of your silo. What height and radius should the silo have for a given volume V?

Although it is possible to solve this problem by the same method used in the can design question, it turns out to be much simpler to use implicit differentiation to find $\frac{d}{dr}SA$.

Answer this question for:

a) a silo with a circular floor (to keep out gophers) and

b) a silo with no built-in flooring (for use in regions with no gophers).

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$$V = \pi r^2 h + \frac{2}{3}\pi r^3$$

$$h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2}$$

$$= \frac{V}{\pi r^2} - \frac{2}{3} r$$

$$\Rightarrow \frac{dh}{dr} = -\frac{2}{3} - \frac{2V}{\pi r^3}$$

body floor roof

a) $SA = 2\pi r h + \pi r^2 + 2\pi r^2$

$$= 2\pi r h + 3\pi r^2$$

$$\frac{d}{dr} SA = \frac{d}{dr} 2\pi r h + \frac{d}{dr} 3\pi r^2$$

$$= 2\pi\left(1 \cdot h + r\frac{dh}{dr}\right) + 6\pi r$$

$$\Rightarrow \frac{d\,SA}{dr} = 2\pi h + 2\pi r\frac{dh}{dr} + 6\pi r$$

$$= 2\pi\left(\frac{V}{\pi r^2} - \frac{2}{3}r\right) + 2\pi r\left(-\frac{2}{3} - \frac{2V}{\pi r^3}\right) + 6\pi r$$

$$= \frac{2V}{r^2} - \frac{4\pi}{3}r - \frac{4\pi r}{3} - \frac{4V}{r^2} + 6\pi r$$

$$= \frac{10\pi}{3}r - \frac{2V}{r^2}$$

$$\frac{10\pi}{3}r - \frac{2V}{r^2} = 0$$

$$\frac{10\pi}{3}r^3 = 2V$$

$$r^3 = \frac{6V}{10\pi} = \frac{3V}{5\pi}$$

$$\pi r^2 = \frac{3V}{5r}$$

$$\Rightarrow h = \frac{V}{\frac{3V}{5r}} - \frac{2}{3}r$$

$$= \frac{5r}{3} - \frac{2}{3}r$$

$$= r$$

$$\therefore r = h = \sqrt[3]{\frac{3V}{5\pi}}$$

$$SA = 2\pi rh + 3\pi r^2$$

$$= 2\pi r\left(\frac{3V - 2\pi r^3}{3\pi r^2}\right) + 3\pi r^2$$

$$= \frac{6V - 4\pi r^3}{3r} + 3\pi r^2$$

$$= \frac{6V + 5\pi r^3}{3r}$$

$$= \frac{2V}{r} + \frac{5}{3}\pi r^2$$

$$h = \frac{V}{\pi r^2} - \frac{2}{3}r$$

$$= \frac{3V - 2\pi r^3}{3\pi r^2}$$

$$3V - 2\pi r^3 = 0$$

$$r^3 = \frac{3V}{2\pi}$$

As $r \to 0$, $SA \to \infty$.

As $r \to \infty$, $SA \to \infty$.

$\therefore$ As $r = h = \sqrt[3]{\frac{3V}{5\pi}}$ is the only critical point, SA must be a minimum at this point.

b) $V = \pi r^2 h + \frac{2}{3}\pi r^3$

$h = \frac{V}{\pi r^2} - \frac{2}{3}r$

$SA = 2\pi r h + 2\pi r^2$

$\frac{dh}{dr} = -\frac{2}{3} - \frac{2V}{\pi r^3}$

$\frac{d}{dr}SA = 2\pi h + 2\pi r\frac{dh}{dr} + 4\pi r$

$\Rightarrow \frac{d}{dr}SA = -\frac{2V}{r^2} - \frac{8\pi r}{3} + 4\pi r$

$= \frac{4\pi r}{3} - \frac{2V}{r^2}$

$\frac{d}{dr}SA = 0 \Rightarrow \frac{4\pi r}{3} = \frac{2V}{r^2}$

$r^3 = 2V \times \frac{3}{4\pi}$

$= \frac{3V}{2\pi}$

$\Rightarrow r = \sqrt[3]{\frac{3V}{2\pi}}$

$\pi r^2 = \frac{3V}{2r}$

$\Rightarrow h = \frac{V}{\frac{3V}{2r}} - \frac{2}{3}r$

$= \frac{2r}{3} - \frac{2}{3}r$

$= 0$

$\therefore$ The minimum SA is a hemisphere with $r = \sqrt[3]{\frac{3V}{2\pi}}$ with no cylinder body.

$SA = 2\pi r(0) + 2\pi r^2$

$= 2\pi r^2$.